HOW TO CALCULATE THE FEDOSOV STAR–PRODUCT
(EXERCICES DE STYLE)

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To the memory of Mosh Flato

Abstract

This is an expository note on Fedosov's construction of deformation quantization. Given a symplectic manifold and a connection on it, we show how to calculate the star-product step by step.
We draw simple diagrams to solve the recursive equations for the Fedosov connection and for flat sections of the Weyl algebra bundle corresponding to functions.
We also reflect on the differences of symplectic and Riemannian geometries.

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1. Deformation quantization of a symplectic manifold

1.1. Fedosov's idea: Koszul–type resolution.

We consider a deformation quantization of a symplectic manifold (M, ω_0) as a deformation of an algebra of smooth functions on M in the direction of the Poisson bracket [1].

Definition 1.1. Deformation quantization of a symplectic manifold (M, ω_0) is an associative algebra structure on $\mathbb{A} = C^\infty(M)[[t]]$ over $\mathbb{C}[[t]]$, called a $*$-product, such that for any $a = a(x,t) = \sum_{k=0}^\infty t^k a_k(x)$ and $b = b(x,t) = \sum_{k=0}^\infty t^k b_k(x),\ a_k(x), b_k(x) \in C^\infty(M)$

1. The product $*$ is local, that is in the $*$-product $a(x,t) * b(x,t) = \sum_{k=0}^\infty t^k c_k(x)$, the coefficients $c_k(x)$ depend only on a_i, b_j and their derivatives $\partial^\alpha a_i, \partial^\beta b$ with $i + j + |\alpha| + |\beta| \leq k$.
2. It is a formal deformation of the commutative algebra $C^\infty(M):\ c_0(x) = a_0(x) b_0(x)$.
3. Let $\{\cdot, \cdot\}$ be the Poisson bracket of functions, given by a bivector field dual to the form ω_0. There is a correspondence principle:
$$[a,b] := \frac{i}{t}(a * b - b * a) = \{a_0(x), b_0(x)\} + t\ r(a,b),$$
where $r(a,b) \in \mathbb{A}$.
4. There is a unit: $a(x,t) * 1 = 1 * a(x,t) = a(x,t)$.

Fedosov found a geometric way to perform the deformation quantization [4, 5] (also see [9] for a comprehensive exposition). The following idea lies behind Fedosov's construction — a Koszul–type

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resolution is considered for $C^\infty(M)[[t]]$:

$$C^\infty(M)[[t]] \xrightarrow{Q} A^0 \xrightarrow{D} A^1 \xrightarrow{D} A^2 \xrightarrow{D} \dots . \tag{1}$$

It means that cohomology groups of this complex are all zero, and in particular we get $C^\infty(M)[[t]] := ImQ \cong KerD$. If each term of the resolution has a noncommutative associative structure and moreover D respects this structure, then it provides the space $C^\infty(M)[[t]]$ with a new associative noncommutative product. Namely, let $\circ : A^0 \otimes A^0 \to A^0$ be such a product on A^0. Then we get a product on $C^\infty(M)[[t]]$ as follows:

$$a * b = Q^{-1}\big(Q(a) \circ Q(b)\big). \tag{2}$$

(D respects the product, so since D is zero on $Q(a)$ and $Q(b)$, it must be zero on $Q(a) \circ Q(b)$, so the product is in the kernel of D, that is in the image of Q, and its preimage Q^{-1} is well defined.) The product on A^0 should also verify certain properties to certify the axioms of the deformation quantization, so it is a very special resolution.

Fedosov constructs such a resolution by using the differential forms on the manifold with values in the Weyl algebra bundle (with Moyal-Vey fibrewise product) [4]. The main step then is to find the differential on it, D, which would respect the algebra structure. This differential is called Fedosov connection and is obtained by an iteration procedure from a torsion free symplectic connection on the manifold.

1.2. **Weyl algebra of a vector space.**

Let E be a vector space with a non–degenerate skew-symmetric form ω. The algebra of polynomials on E is the algebra of symmetric powers of E^*, $S(E^*)$, and it has a skew-symmetric form on it which is dual to ω. Let e be a point in E and $\{e^k\}$ denote its linear coordinates in E with respect to some fixed basis. Then $\{e^k\}$ define a basis in E^*. Let ω^{kl} be the matrix for the skew-symmetric form on E^*. Let us consider the power series in t with values in $S(E^*)$:

Definition 1.2. The Weyl algebra $W(E^*)$ of a vector space E^* is an associative algebra

$$W(E^*) = S(E^*)[[T]] : \quad a(e,t) = \sum_{k \geq 0} a_k(e) t^k,$$

with the product structure given by the Moyal–Vey product:

$$a \circ b(e,h) = \exp\, \{-\frac{it}{2}\omega^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial z^l}\} \;\; a(x,t)\; b(z,t)\, \Big|_{x=z=e}. \tag{3}$$

The Lie bracket is defined with respect to this product. We can look at this algebra as at a completion of the universal enveloping algebra of the Heisenberg algebra on $E^* \oplus t\mathbb{C}$, namely, the algebra with relations

$$e^k \circ e^l - e^l \circ e^k = -it\omega^{kl} \tag{4}$$

where $\omega^{kl} = \omega(e^k, e^l)$ defines a Poisson bracket on E^*.

Let us consider the product of the Weyl algebra and the exterior algebra of the space E^*: $W(E^*) \otimes \Lambda E^*$. Let dx^k be the basis in ΛE^* corresponding to e^k in $W(E^*)$.

There is a decreasing filtration on the Weyl algebra $W(E^*)$: $W_0 \supset W_1 \supset W_2 \supset$ ... given by the degree of generators. The generators e's have degree 1 and t has degree 2:

$$W_p = \{\text{elements with degree} \geq p\}.$$

One can define a grading on W as follows

$$gr_i W = \{\text{elements with degree} = i\}.$$

it is isomorphic to W_i/W_{i+1}. One can see that the product (3) preserves the grading (since the relation (4) is homogeneous).

Definition 1.3. An operator on $W(E^*) \otimes \Lambda E^*$ is said to be of degree k if it maps $W_i \otimes \Lambda E^*$ to $W_{i+k} \otimes \Lambda E^*$ for all i.

Such an operator defines maps $gr_i W \otimes \Lambda E^*$ to $gr_{i+k} W \otimes \Lambda E^*$ for all i.

Definition 1.4. Derivation on $W(E^*) \otimes \Lambda E^*$ is a linear operator which satisfies the Leibnitz rule:

$$D(ab) = (Da)b + (-1)^{\tilde{a}\tilde{D}} a(Db)$$

where $\tilde{a}$ and $\tilde{D}$ are corresponding degrees. It turns out that all $\mathbb{C}[[t]]$–linear derivations are inner [2].

Lemma 1.5. *Any linear derivation D on $W(E^*) \otimes \Lambda E^*$ is inner, namely there exists such $v \in W(E^*)$ so that $Da = \frac{i}{t}[v, a]$ for any $a \in W(E^*)$*

Proof. Indeed, for the generators $\frac{\partial}{\partial e^k} a = \frac{i}{2t}[\omega_{kl} e^l, a]$. So for any derivation one can get a formula: $Da = \frac{i}{t}[\frac{1}{2}\omega_{kl} e^k D e^l, a]$. ∎

One can define two natural operators on the algebra $W(E^*) \otimes \Lambda E^*$: δ and δ^* of degree -1 and 1 correspondingly, such that δ is the lift of the "identity" operator

$$u : e^k \otimes 1 \to 1 \otimes dx^k$$

and δ^* is the lift of its inverse. On monomials $e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ... \wedge dx^{j_n} \in W^m(E^*) \otimes \Lambda^n E^*$ δ and δ^* can be written as follows:

$$\begin{aligned}
\delta :\ & e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ... \wedge dx^{j_n} \mapsto \\
& \sum_{k=1}^{m} e^{i_1} \otimes ... \widehat{e^{i_k}} ... \otimes e^{i_m} \otimes dx^{i_k} \wedge dx^{j_1} \wedge ... \wedge dx^{j_n} \\
\delta^* :\ & e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ... dx^{j_n} \mapsto \\
& \sum_{l=1}^{n} (-1)^l e^{j_l} \otimes e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ... \widehat{dx^{j_l}} ... \wedge dx^{j_n}.
\end{aligned}$$

Lemma 1.6. *Operators δ and δ^* have the following properties:*

$$\delta a = dx^l \frac{\partial a}{\partial e^l} = [-\frac{i}{t}\omega_{kl}\ e^k dx^l,\ a], \quad \delta^* a = y^l \iota_{\frac{\partial}{\partial x^l}} a, \quad \delta^2 = {\delta^*}^2 = 0$$

On monomials $e^{i_1} \otimes ... \otimes e^{i_m} \otimes dx^{j_1} \wedge ... \wedge dx^{j_n}$ from $gr_m W(E^*) \otimes \Lambda^n E^*$

$$\delta\delta^* + \delta^*\delta = (m+n) Id,$$

where Id is the identity operator. Any element $a \in gr_m W(E^*) \otimes \Lambda^n E^*$ has a decomposition:

$$a = \frac{1}{m+n}(\delta\delta^* a + \delta^*\delta a) + a_0.$$

where a_0 is a projection of $a \in W(E^*) \otimes \Lambda E^*$ to the center of the algebra, that is the summands in a which do not contain e-s.

1.3. **Symplectic connections (symplectic covariant derivatives).**

The term *symplectic connection* in this section is in fact a *symplectic covariant derivative.*

Let us consider connections on a manifold M.

Proposition 1.7. *Let ω be a skew-symmetric 2-form on $\mathcal{T}M$. Let ∇ be a torsion free connection preserving this form. Then ω is necessarily closed.*

Proof. The skew-symmetry of ω is the following condition: $\omega(X,Y) = -\omega(Y,X)$. The connection ∇ is torsion–free when $\nabla_X Y - \nabla_Y X = [X,Y]$. Suppose such ∇ exists. The connection ∇ preserves the form ω when $\nabla\omega = 0$. This means that for all $X, Y, Z \in \mathcal{T}M$:

$$\nabla_X(\omega(Y,Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) \tag{5}$$

Since $\omega(Y,Z)$ is a function $\nabla_X(\omega(Y,Z)) = X\omega(Y,Z)$. Then,

$$\begin{aligned} X\omega(Y,Z) \quad - \quad Y\omega(X,Z) \quad + \quad Z\omega(X,Y) & \\ = \quad & \omega(\nabla_X Y, Z) - \omega(\nabla_X Z, Y) - \omega(\nabla_Y X, Z) \\ & + \omega(\nabla_Y Z, X) + \omega(\nabla_Z X, Y) - \omega(\nabla_Z Y, X) \\ = \quad & \omega([X,Y],Z) - \omega([X,Z],Y) + \omega([Y,Z],X) \end{aligned}$$

which is exactly the condition $d\omega = 0$. ■

Remark 1.8. Here we want to make an analogy with a Riemannian case. The Riemannian metric is a symmetric two-form and there is a unique torsion free connection compatible with it (the Levi–Civita connection).

In symplectic case we deal with a skew-symmetric form. There are many different torsion free connections preserving the form if the form is closed.

The statement of uniqueness of Levi–Civita connection in the Riemannian case is substituted by the requirement for the form to be closed in the skew-symmetric setting:

- Symmetric: Torsion–free compatible connection always exists and unique (Levi–Civita connection).
- Skew-symmetric: Torsion–free compatible connection exists if the form is closed and, in general, not unique.

Here we are mostly interested in the case when M is a symplectic manifold, that is when there is a symplectic form ω on M (a closed and nondegenerate 2-form on $\mathcal{T}M$).

Definition 1.9. A connection which preserves a symplectic form is called a symplectic connection.

Any connection on a symplectic manifold gives rise to a symplectic connection:

Proposition 1.10. [10] [11]. *Let (M,ω) be a symplectic manifold. Then for every connection ∇ there exists a three–tensor S, such that*

$$\tilde{\nabla} = \nabla + S$$

is a symplectic connection.

Then for $X, Y \in \mathcal{T}M$

$$\hat{\nabla}_X Y = \tilde{\nabla}_X Y - \frac{1}{2} Tor(X,Y)$$

defines a torsion–free connection compatible with the form ω. Here 2–form Tor is the torsion of $\tilde{\nabla}$

$$Tor(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - \tilde{\nabla}_{[X,Y]}$$

Then S is defined as follows:

$$S_X Y = \frac{1}{2}\{(\nabla_X \omega)(Y, .)\}^{\sharp},$$

where $\sharp : \mathcal{T}^*M \to \mathcal{T}M$ is the inverse to $\flat$, given by :

$$\flat : \mathcal{T}M \to \mathcal{T}^*M$$
$$u^{\flat} = \omega(u, .) \ for \ \ u \in \mathcal{T}M.$$

Symplectic connections form an affine space with the associated vector space $\mathcal{A}^1(M, sp(2n))$, the Lie algebra $sp(2n)$–valued one-forms on M.

1.4. **Weyl algebra bundle and Fedosov's theorem.**

Let M^{2n} be a symplectic manifold with a symplectic form ω. In local coordinates at a point x:

$$\omega = \omega_{kl} dx^k \wedge dx^l.$$

The symplectic form on a manifold M defines a Poisson bracket on functions on M. For any two functions $a, b \in C^{\infty}(M)$:

$$\{a, b\} = \omega^{kl} \frac{\partial a}{\partial x^k} \frac{\partial b}{\partial x^l} \tag{6}$$

where $(\omega^{kl}) = (\omega_{kl})^{-1}$ (as matrix coefficients).

We can define the bundle of Weyl algebras $\mathcal{W} \to M$, with the fibre at a point $x \in M$ being the Weyl algebra of the vector space $\mathcal{T}^*_x M$. Let $\{e^1, ...e^{2n}\}$ be $2n$ generators in $\mathcal{T}^*_x M$, corresponding to dx^k. The form ω^{kl} defines a pointwise Moyal–Vey product.

The filtration and the grading in $\mathcal{W}$ are inherited from $W(\mathcal{T}^*_x M)$ at each point $x \in M$. We denote by $\mathcal{W}^k$ the k-th graded component in $\mathcal{W}$:

$$\mathcal{W} = \oplus_i \mathcal{W}^k$$

A symplectic connection, ∇, satisfying (5) can be naturally lifted to act on any symmetric power of the cotangent bundle (by the Leibniz rule). Moreover, since the cotangent bundle $\mathcal{T}^*M \cong \mathcal{W}^1$, we can lift ∇ to be an operator on sections $\Gamma(M, \mathcal{W}^k)$ with values in $\Gamma(M, \mathcal{T}^*M \otimes \mathcal{W}^k)$. By abuse of notations this operator is also called ∇.
The connection ∇ preserves the grading, in other words it is an operator of degree zero. It is clear that in general this connection is not flat: $\nabla^2 \neq 0$. Fedosov's idea is that for $\mathcal{W}_M$ bundle one can add to the initial symplectic connection some operators not preserving the grading so that the sum gives a flat connection on the Weyl bundle.

Theorem 1.11. *(Fedosov.) There is a unique set of operators* $r_k : \Gamma(M, \mathcal{W}^k) \to \Gamma(M, \mathcal{T}^*M \otimes \mathcal{W}^{i+k})$ *such that*

$$D = -\delta + \nabla + r_1 + r_2 + \dots \tag{7}$$

$$D^2 = 0, \ \textit{and} \ \delta^* r_i = 0.$$

There is a one-to-one correspondence between formal series in t *with coefficients in smooth functions* $C^{\infty}(M)$ *and horizontal sections of this connection:*

$$Q : C^{\infty}(M)[[t]] \to \Gamma_{flat}(M, \mathcal{W}_M). \tag{8}$$

Main idea of the proof is to use the following complex:

$$0 \to \Gamma(M, \mathcal{W}) \xrightarrow{\delta} \mathcal{A}^1(M, \mathcal{W}) \xrightarrow{\delta} \mathcal{A}^2(M, \mathcal{W}) \xrightarrow{\delta} \dots , \tag{9}$$

where $\mathcal{A}^n(M, \mathcal{W})$ denotes C^∞–sections of n–form bundle with values in the bundle $\mathcal{W}$,

$$\mathcal{A}^k(M, \mathcal{W}) = \Gamma(M, \Lambda^k \mathcal{T}^* M \otimes \mathcal{W}).$$

This complex is exact since δ is homotopic to identity by δ^*. For each i the equation for r_i has the form

$$\delta(r_i) = \text{function}(\nabla, r_1, \dots , r_{i-1}). \tag{10}$$

However it is not difficult to show that this function is in the kernel of δ hence r_i exists.

The noncommutative associative structure on the Weyl bundle determines a $*$–product on functions by the correspondence 2.

In fact, the equation $D^2 = 0$ is just the Maurer–Cartan equation for a flat connection. One can see the analogy with Kazhdan connection [6] on the algebra of formal vector fields. Notice that $\delta = dx^k \frac{\partial}{\partial e^k}$ is of degree -1. The flatness of the connection is given by the recurrent procedure, namely starting from the terms of degree -1 and 0 one can get other terms step by step. While Kazhdan connection does not have a parameter involved it has the same structure – it starts with known -1 and 0 degree terms. Other terms are of higher degree and can be recovered one by one.

Let us also mention here that the connection D can be written as a sum of two terms – one is a derivation along the manifold, the usual differential d, and the other is an endomorphism of a fibre, let us call it Γ. Since all endomorphisms are inner one can write it as an adjoint action with respect to the Moyal product. Γ acts adjointly by an operator from $\Gamma(M, \mathcal{W})$ to $\Gamma(M, \mathcal{T}^* M \otimes \mathcal{W})$.

$$D = d + \Gamma = d + \frac{i}{t}[\gamma, \cdot]_\circ, \tag{11}$$

where the Lie bracket is the commutator bracket, and $\gamma \in \Gamma(M, \mathcal{T}^* M \otimes \mathcal{W})$. Then the equation $D^2 = 0$ becomes:

$$d\Gamma + \frac{1}{2}[\Gamma, \Gamma]_\circ = 0.$$

The same equation for γ then is as follows:

$$\omega + d\gamma + \frac{i}{t}\frac{[\gamma, \gamma]_\circ}{2} = 0, \tag{12}$$

where ω is a central 2–form. This equation states that D^2 is given by an adjoint action of a central element, so it is zero. However it turns out to be very important which exactly form ω is given in the center by the connection D. Inner automorphisms of the Weyl algebra are given by the adjoint action by elements of the algebra (Lemma 1.5). Its central extension gives the whole algebra. Curvature of Fedosov connection is zero, however its lift to the central extension is nonzero.

Definition 1.12. The characteristic class of the deformation is the cohomology class of the form $[\omega] \in H^2(M)[[t]]$.

2. Calculations from diagrams.

Fedosov quantization produces the system of recursively defined equations in order to find the flat connection and then another system for the flat sections of this connection. Fedosov proved that there are no obstructions to solutions. In what follows we show how to obtain these equations step by step from diagrams, which present all equations at once. Since there are other situations when

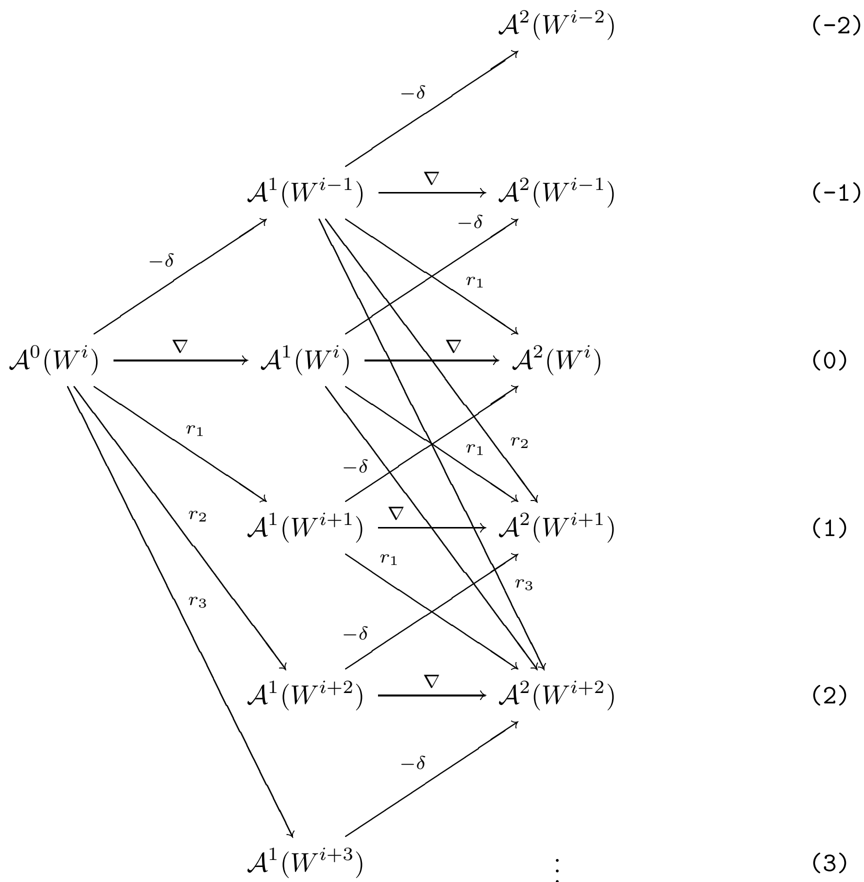
one has to solve systems of recursive equations on graded objects we also hope that our presentation might be useful in some other calculations possibly of completely different origin.

2.1. **Fedosov connection.**
Let $\mathcal{A}^p(W^i) = \Gamma(M, \Lambda^p \mathcal{T}^* M \otimes \mathcal{W}^i)$. Let us represent the action of

$$D = -\delta + \nabla + r_1 + r_2 + r_3 + r_4 + \dots$$

by arrows pointing in directions corresponding to the degree of each component.

Namely, $r_k : \mathcal{A}^0(W^i) \to \mathcal{A}^1(W^{i+k})$ is drawn to go from the point corresponding to the level i in the first column to the point in the second column k rows down: δ goes up one row, ∇ is on the same level, r_1 goes down one level and so on. Same operators act between first and second column.



Key observation. The curvature is equal to 0 if the connection applied twice to any element $a \in \Gamma(M, \mathcal{W})$: $D^2 a = 0$ is 0 in each degree. In other words: the sum of arrows coming to the second column $\mathcal{A}^2(W^i)$ should be 0 for every i.

Showing that this is true for any element in $\mathcal{A}^0(W^i) = \Gamma(M, \mathcal{W}^i)$ for any i will do.

First two terms, δ of degree -1 and ∇ of degree 0, are known, our purpose is to find the other terms recursively.
For every degree i we get equations on operators:

- Level (-2): $\delta^2 = 0$
- Level (-1): $-[\delta, \nabla] = 0$

- Level (0): $-[\delta, r_1] + \nabla^2 = 0$
- Level (1): $-[\delta, r_2] + [\nabla, r_1] = 0$
- Level (2): $-[\delta, r_3] + [\nabla, r_2] + \frac{[r_1, r_1]}{2} = 0$
- Level (3): $-[\delta, r_4] + [\nabla, r_3] + [r_1, r_2] = 0$

These equations are the graded components of

$$D^2 = (-\delta + \nabla + [r, .])^2 = \nabla^2 - [\delta, r] + [\nabla, r] + r^2 = 0$$

where $r = r_1 + r_2 + \dots$ It is solved recursively: in each degree $k \geq 0$ one gets an equation involving only r_i with $i \leq k$.
Let us show what happens in the first few equations.

Degree -2. The equation is $\delta^2 = 0$. It is satisfied by the Lemma (1.6).

Degree -1. Next one is $[\delta, \nabla] = 0$. It is true by a simple calculation.
For this equation we need that the connection ∇ is torsion–free.

Degree 0. Here is the first nontrivial calculation. We have to find such r_1 that $-[\delta, r_1] = \nabla^2$.
a) Existence. First of all:

$$[\delta, \nabla^2] = [\delta, \nabla]\nabla - \nabla[\delta, \nabla]$$

which is 0 by the previous equation.
There is an operator δ^* which is a homotopy for δ.

$$\delta^*\delta^* = 0, \quad \delta\delta^* + \delta^*\delta = id\ c \tag{13}$$

This c is a number of y 's and dx 's, for example for a term $y^{i_1} \dots y^{i_p} dx^{j_1} \dots dx^{j_q}$ this number $c = p + q$. Let us put

$$r_1 = \delta^* \nabla^2$$

then indeed:

$$[\delta, r_1] = \delta(r_1) = \delta(\delta^* \nabla^2) = \nabla^2$$

and also $\delta^* r_1 = 0$.
b) Uniqueness.
Let $r_1' = r_1 + \alpha$, such that $\delta^*\alpha = 0$. Then $\alpha = \delta^*\beta$ for some β. Hence, $\delta\delta^*\beta = 0$, because $\delta(r_1 + \alpha) = \delta r_1$.
From (13) we get that $\beta = \delta^*\delta\beta$ and $\alpha = \delta^*\beta = \delta^*(\delta^*\delta\beta) = 0$

Degree 1. $[\delta, r_2] = [\nabla, r_1]$ gives the equation on the operator r_2.
a) Existence. Again we show

$$[\delta, [\nabla, r_1]] = [[\delta, \nabla], r_1] - [\nabla, [\delta, r_1]] = 0$$

b) Uniqueness. $r_2 = \delta^*([\nabla, r_1])$ similar to the previous one.
Recursively getting similar equations for r_n one finds the Fedosov connection. Here are first few terms:

$$D = \delta + \nabla + \delta^{-1}\nabla^2 + \delta^{-1}\{\nabla, \delta^{-1}\nabla^2\} + \dots$$

2.2. Flat sections and the $*$–product.

Series in t with functional coefficients are in one-to-one correspondence with flat sections of Fedosov connection:

$$Q : C^\infty(M)[[t]] \to \mathcal{A}^0(W)$$

Fedosov connection maps in a unique way each series

$$a = a_0 + ta_1 + t^2a_2 + \dots$$

to a flat section of the Weyl algebra bundle

$$A = a + A_1 + A_2 + A_3 + \dots$$

verifying an equation:

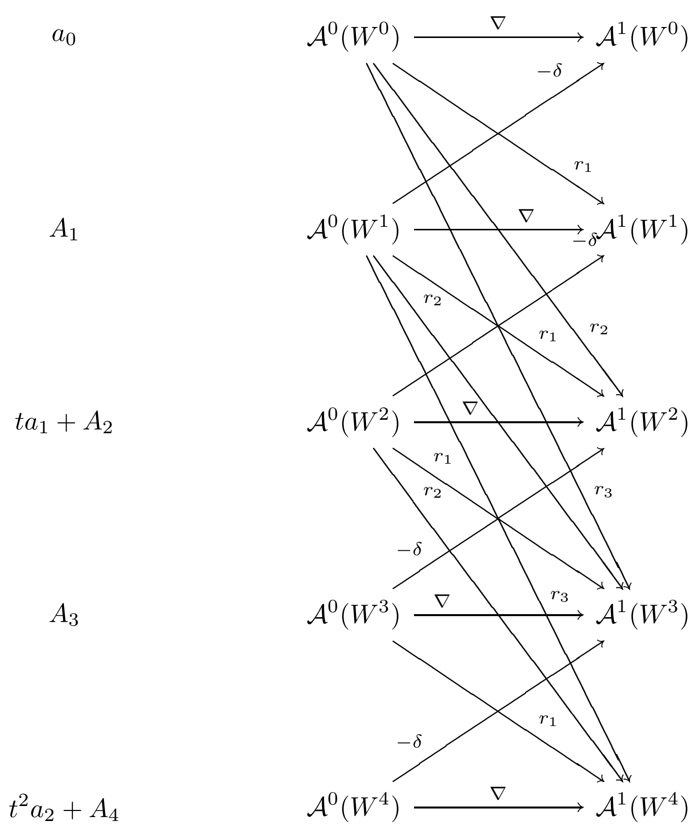
$$\delta^{-1}(A - a) = 0.$$

This last condition makes the operator $Q^{-1} : \mathcal{A}^0(W) \to C^\infty(M)[[t]]$ simple, namely it is just an evaluation of $A \in \mathcal{A}^0(W)$ at zero value of coordinates along the fibres W. This condition could be changed for any other condition fixing the zero section in $\mathcal{A}^0(W)$ (see [3]).

The condition of flatness:

$$DA = 0$$

can be represented by the fact that for all i sum of operators r_k which get to $\mathcal{A}^1(W^i)$ must be 0. It again gives a recursive system of equations.



We notice that all r_k kill functions, because r_k acts as adjoint operators and functions are in the center of $\mathcal{A}^0(W)$, so $r_k(a_i) = 0$. Hence first few equations following from the diagram above are

1. $\nabla a_0 - \delta A_1 = 0$

2. $\nabla A_1 - \delta(ta_1 + A_2) = 0$
3. $r_1 A_1 + \nabla(ta_1 + A_2) - \delta A_3 = 0$
4. $r_2 A_1 + r_1 A_2 + \nabla A_3 - \delta(t^2 a_2 + A_4) = 0$
5. $r_3 A_1 + r_2 A_2 + r_1 A_3 + \nabla(t^2 a_2 + A_4) - \delta A_5 = 0$

Let dx^k be a local frame in $\mathcal{T}^*M$. Then let the corresponding generators in W be $\{e^k\}$. Then A_i are of the form $A_{k_1\dots k_i} e^{k_1} \dots e^{k_i}$.

The symplectic connection ∇ locally can be written as:

$$\nabla = dx^l \frac{\partial}{\partial x^l} + \frac{i}{2t}[\Gamma_{jkl} dx^l e^k e^l, \quad]$$

So for the first terms of the flat section corresponding to $a = a_0$ we get:

1. $A_1 = \delta^{-1}\nabla a = \delta^{-1} da = \partial_l a e^l$
2. $A_2 = \delta^{-1}\nabla A_1 = \delta^{-1}\nabla(\partial_l a e^l) = \{\partial_k \partial_l a + \Gamma_{kl}{}^l \partial_j a\} e^k e^l$

The first few terms in the $*$–product of two functions $a, b \in C^\infty(M)$ are:

$$a * b = ab - \frac{it}{2}\omega^{kl}\partial_i a \partial_j b - t^2(\partial_i\partial_j a + \Gamma^l_{kl}\partial_l a)\omega^{im}\omega^{jn}(\partial_m\partial_n b + \Gamma^k_{mn}\partial_k b) + \dots$$

2.3. **Standard example of deformation quantization.**

The procedure of deformation quantization requires calculations which are not obvious and most of the time do not give nice formulas. However in few cases for particular manifolds one can calculate the $*$–product explicitly. The first trivial example is the quantization of $\mathbb{R}^{2n}$ with a standard symplectic form. Let $\{x^1, \dots, x^{2n}\}$ be a local coordinate system at some point $x \in \mathbb{R}^{2n}$. The Darboux symplectic form in these coordinates is

$$\omega = dx^i \wedge dx^{i+n}, \quad 1 \le i \le n$$

The standard symplectic form and the trivial connection $\nabla = d$ give an algebra of differential operators in $\mathbb{R}^{2n}$. Namely, using calculations from the diagrams we get the Fedosov connection to be:

$$D = d - \delta \quad \text{or in coordinates} \quad D = dx^k\left(\frac{\partial}{\partial x^k} - \frac{\partial}{\partial e^k}\right).$$

Flat section of such a connection corresponding to a function a under the quantization map is as follows

$$A = a + e^k \frac{\partial a}{\partial x^k} + e^k e^l \frac{\partial^2 a}{\partial x^k \partial x^l} + \cdots .$$

We see that it gives a formula for Taylor decomposition of a function a at a point x. In fact the e^k terms can be considered as jets. Then the $*$–product of two flat sections is given by the formula (3). It is easy to deduce that for two functions a and b the $*$–product is

$$\begin{aligned} a * b &= \exp\{-it\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^{n+k}}\} a(y)b(z)|_{y=z=x} \\ &= ab - it\frac{\partial a}{\partial x^k}\frac{\partial b}{\partial x^{n+k}} + \frac{t^2}{2}\left(\frac{\partial^2 a}{\partial x^k \partial x^l}\right)\left(\frac{\partial^2 b}{\partial x^{n+k}\partial x^{n+l}}\right) + \cdots . \end{aligned} \tag{14}$$

Let us map $C^\infty(\mathbb{R}^{2n})$ to differential operators on $\mathbb{R}^n$, considered as polynomials on $T^*\mathbb{R}^n$. Then the $*$–product gives exactly the product of differential symbols.

Remark 2.1. The same scheme actually works for any cotangent bundle $\mathcal{T}^*M$ with the canonical symplectic form – the quantized algebra of functions on $\mathcal{T}^*M$ is isomorphic to the algebra of differential operators on M. This observation leads to various types of index theorems [8], also [7],[12].

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